A representation theorem for independence algebras

K. Urbanik

Independence algebras, also known as v^*-algebras

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We say that $\langle A, \mathbf{A} \rangle$ satisfies the **exchange property** (EP), if for every subset X of A and all elements $x, y \in A$ if

$$y \in \langle X \cup \{x\} \rangle \text{ and } y \notin \langle X \rangle$$

then $x \in \langle X \cup \{y\} \rangle$.

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(ii) A subset X is a basis if and only if X is a minimal generating set if and only if X is the maximal independent set.
(iii) All of bases of A has the same cardinality, called the **dimension** of A.

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We say that a mapping θ from A into itself is an **endomorphism** if for any n-ary term operation $t(x_1, \cdots, x_n)$ we have

$$t(x_1, \cdots, x_n)\theta = t(x_1\theta, \cdots, x_n\theta).$$

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An algebra $\langle A, \mathbf{A} \rangle$ satisfying the exchange property is called an **independence algebra** if it satisfies the **free basis property**, by which we mean that for any basis X of A and a map $\alpha : X \longrightarrow A$, α can be extended to an endomorphism of A.

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For any independence algebra A, we have

$$\langle \emptyset \rangle = C$$

where C is the collection of all elements $u \in A$ such that there is a constant term operation $t(x_1, \cdots, x_n)$ of A whose image is u.

v-algebras

We say that an equality

$$f(x_1, \cdots, x_n) = g(x_1, \cdots, x_n)$$

depends on x_j $(1 \leq j \leq n)$, if there exists a system $a_1, \cdots, a_n, a'_j$ of elements belonging to A for which

$$f(a_1, \cdots, a_{j-1}, a_j, a_{j+1}, \cdots, a_n) = g(a_1, \cdots, a_{j-1}, a_j, a_{j+1}, \cdots, a_n)$$

and

$$f(a_1, \cdots, a_{j-1}, a'_j, a_{j+1}, \cdots, a_n) \neq g(a_1, \cdots, a_{j-1}, a'_j, a_{j+1}, \cdots, a_n).$$

v-algebras

An algebra $\langle A, \mathbf{A} \rangle$ is called a v-**algebra** if for every pair of integers j, n $(1 \leq j \leq n)$ and for every pair of n-ary term operations for which the equality

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depends on x_j $(1 \leq j \leq n)$, there exists a $(n-1)$-ary term operation h such that the above equality is equivalent to the equality

$$x_j = h(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n).$$

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Note v-algebras are included in v^*-algebras.

Representation theorem

Let $\mathcal{G}$ be a group of transformations of a non-empty set A. We say that a subset $B \subseteq A$ is **normal** with respect to the group $\mathcal{G}$ if B contains fixed points of all transformations that are not the identity belonging to $\mathcal{G}$ and $g(B) \subseteq B$ for every $g \in \mathcal{G}$.

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$\mathbf{A}^{(0)}$: the class of all values of constant term operations of A.

$\mathbf{A}^{(n)}$: the class of all n-ary term operations of A, where $n \geq 1$.

$\mathbf{A}^{(n,k)}$: the subclass of $\mathbf{A}^{(n)}$ containing all n-ary term operations depending on at most k variables. i.e. $f \in \mathbf{A}^{(n,k)}$ if there is some $g \in \mathbf{A}^{(k)}$ such that

$$f(x_1, \cdots, x_n) = g(x_{i_1}, \cdots, x_{x_{i_k}})$$

for a system of indices $i_1, \cdots, i_k$ and for every $x_1, \cdots, x_n \in A$.

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Representation theorem

Let $\langle A, \mathbf{A} \rangle$ be a v-algebra. Then one of the following holds:
(i) If $\mathbf{A}^{(0)} \neq \emptyset$ and $\mathbf{A}^{(3)} \neq \mathbf{A}^{(3,1)}$, then there is a field $\mathcal{K}$ such that A is a linear space over $\mathcal{K}$ and, further, there exists a linear subspace A_0 of A such that $\mathbf{A}$ is the class of all term operations f defined as

$$f(x_1, \cdots, x_n) = \sum_{k=1}^{n} \lambda_k x_k + a,$$

where $\lambda_1, \cdots, \lambda_k \in \mathcal{K}$ and $a \in A_0$.

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where $\lambda_1, \cdots, \lambda_k \in \mathcal{K}$ and $a \in A_0$.
(ii) If $\mathbf{A}^0 = \emptyset$ and $\mathbf{A}^{(3)} \neq \mathbf{A}^{(3,1)}$, then there is a field $\mathcal{K}$ such that A is a linear space over $\mathcal{K}$ and further, there exits a linear subspace A_0 of A such that $\mathbf{A}$ is the class of all term operations f defined as

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where $\lambda_1, \cdots, \lambda_k \in \mathcal{K}, \sum_{k=1}^{n} \lambda_k = 1$ and $a \in A_0$.

Representation theorem

(iii) If $\mathbf{A}^{(3)} = \mathbf{A}^{(3,1)}$, then there is a group $\mathcal{G}$ of transformations of the set A such that every transformation that is not the identity has at most one fixed point in A. Moreover, there is a subset $A_0 \subseteq A$ normal with respect to the group $\mathcal{G}$ such that $\mathbf{A}$ is the class of all term operations f defined as

$$f(x_1, \cdots, x_n) = g(x_j) \quad (1 \leq j \leq n),$$

or

$$f(x_1, \cdots, x_n) = a,$$

where $g \in \mathcal{G}$ and $a \in A_0$.
Note In case (iii), we have $\mathbf{A}^{(n)} = \mathbf{A}^{(n,1)}$.

Representation theorem

Let $\langle A, \mathbf{A} \rangle$ be a v^*-algebra with dimension at least three. Then one of the following cases holds.
(i) $\langle A, \mathbf{A} \rangle$ is a v-algebra.
(ii) There exist a permutation group $\mathcal{G}$ of the set A and a subset A_0 of A normal with respect to $\mathcal{G}$ such that $\mathbf{A}$ is the class of all term operations f defined as

$$f(x_1, \cdots, x_n) = g(x_j) \quad (1 \le j \le n),$$

or

$$f(x_1, \cdots, x_n) = a,$$

where $g \in \mathcal{G}$ and $a \in A_0$.
Note In the above case (ii), $\mathbf{A}^{(n)} = \mathbf{A}^{(n,1)}$.

Main ideas

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We say that a term operation $s \in \widetilde{\widetilde{\mathbf{A}}}^{(3)}$ is **quasi-symmetric** if

$$s(x_1, x_2, x_1) = s(x_2, x_1, x_1) = x_2$$

for each $x_1, x_2 \in A$.

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Fact 2: If s is a quasi-symmetric term operation, then for all $x_1, x_2, x_3, x_4 \in A$ the following equalities are true:
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(iii) $f(s(x_1, x_2, x_3), x_3) = s(f(x_1, x_3), f(x_2, x_3), x_3)$ for any $f \in \widetilde{\mathbf{A}}^{(2)}$.

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(iii) $f(s(x_1, x_2, x_3), x_3) = s(f(x_1, x_3), f(x_2, x_3), x_3)$ for any $f \in \widetilde{\mathbf{A}}^{(2)}$.
(iv) $f(x_1, x_2, x_3) = s(f(x_1, x_1, x_3), f(x_1, x_2, x_1), x_1)$ for any $f \in \widetilde{\mathbf{A}}^{(3)}$.

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Fact 4: If $\mathbf{A}^{(3)} \neq \mathbf{A}^{(3,1)}$, then $\mathcal{K}$ is a field with respect to the operations:

$$(\lambda + \mu)(x_1, x_2) = s(\lambda(x_1, x_2), \mu(x_1, x_2), x_2),$$

$$(\lambda \cdot \mu)(x_1, x_2) = \lambda(\mu(x_1, x_2), x_2),$$

where s is a quasi-symmetric term operation.

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where s is a quasi-symmetric term operation.
Note: The zero element and the unit element of $\mathcal{K}$ are defined by

$$0(x_1, x_2) = x_2, \quad 1(x_1, x_2) = x_1.$$

Main ideas

Fact 5: If $\mathbf{A}^{(3)} \neq \mathbf{A}^{(3,1)}$, then A is a linear space over $\mathcal{K}$ with respect to the operations:

$$x + y = s(x, y, \theta) \quad (x, y \in A)$$

$$\lambda \cdot x = \lambda(x, \theta) \quad (\lambda \in \mathcal{K}, x \in A),$$

where θ is an element of $\mathbf{A}^{(0)}$ if $\mathbf{A}^{(0)} \neq \emptyset$ and is an element of A if $\mathbf{A}^{(0)} = \emptyset$.

Main ideas

Fact 6: If $\mathbf{A}^{(3)} \neq \mathbf{A}^{(3,1)}$, then all term operation f defined as

$$f(x_1, \cdots, x_n) = \sum_{k=1}^{n} \lambda_k x_k,$$

where $\lambda_1, \cdots, \lambda_n \in \mathcal{K}$ and $\sum_{k=1}^{n} \lambda_k = 1$, belong to $\widetilde{\mathbf{A}}^{(n)}$ $(n = 1, 2, \cdots)$.

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Fact 7: If $\mathbf{A}^{(3)} \neq \mathbf{A}^{(3,1)}$, then all term operation f belonging to $\widetilde{\mathbf{A}}^{(n)}$ $(n = 1, 2, \cdots)$ are of the form

$$f(x_1, \cdots, x_n) = \sum_{k=1}^{n} \lambda_k x_k,$$

where $\lambda_1, \cdots, \lambda_n \in \mathcal{K}$ and $\sum_{k=1}^{n} \lambda_k = 1$.

Main ideas

Fact 8: If $\mathbf{A}^{(3)} \neq \mathbf{A}^{(3,1)}$, then the set

$$A_0 = \{f(\theta) : f \in \mathbf{A}^{(1)}\}$$

is a linear subspace of A. Moreover, for every $f \in \mathbf{A}^{(1)}$ there is an element $\lambda \in \mathcal{K}$ such that

$$f(x) = \lambda x + f(\theta)$$

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Fact 9: If $\mathbf{A}^{(3)} = \mathbf{A}^{(3,1)}$, then $\mathbf{A}^{(n)} = \mathbf{A}^{(n,1)}$ for every $n \geq 1$.

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Proof of the Representation Theorem of v-algebras:

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(i) If $\mathbf{A}^{(0)} \neq \emptyset$ and $\mathbf{A}^{(3,1)} \neq \mathbf{A}^{(3)}$, then by **Fact 4** and **Fact 5**, there is field $\mathcal{K}$ such that A is a linear space over $\mathcal{K}$. Taking into account of the definition of addition and scalar-multiplication in A and the definition of θ, we infer that all term operations defined as

$$f(x_1, \cdots, x_n) = \sum_{k=1}^{n} \lambda_k x_k + a,$$

where $\lambda_1, \cdots, \lambda_n \in \mathcal{K}$ and $a \in A_0$, belong to $\mathbf{A}$.

Main ideas

Now, let $f \in \mathbf{A}$. By **Fact 8**, we have the equality

$$\hat{f}(x) = \lambda x + a$$

where $\lambda \in \mathcal{K}$ and $a = f(\theta) \in A_0$. Put

$$g(x_1, \cdots, x_n) = f(x_1, \cdots, x_n) - \lambda x_n - a + x_n.$$

Obviously, $\hat{g}(x) = x$, so that $g \in \widetilde{\mathbf{A}}^{(n)}$. Using **Fact 7**, we have the equality

$$g(x_1, \cdots, x_n) = \sum_{k=1}^{n} \mu_k x_k,$$

where $\mu_1, \cdots, \mu_k \in \mathcal{K}$, and hence we get the representation

$$f(x_1, \cdots, x_n) = \sum_{k=1}^{n} \lambda_k x_k + a.$$

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(ii) If $\mathbf{A}^{(0)} = \emptyset$ and $\mathbf{A}^{(3,1)} \neq \mathbf{A}^{(3)}$, then by **Fact 4** and **Fact 5**, there is a field $\mathcal{K}$ such that A is a linear space over $\mathcal{K}$.

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(ii) If $\mathbf{A}^{(0)} = \emptyset$ and $\mathbf{A}^{(3,1)} \neq \mathbf{A}^{(3)}$, then by **Fact 4** and **Fact 5**, there is a field $\mathcal{K}$ such that A is a linear space over $\mathcal{K}$.
First, it is proved that for all functions f belonging to $\mathbf{A}$ are of the form

$$f(x_1, \cdots, x_n) = g(f_0(x_1, \cdots, x_n))$$

where $g \in \mathbf{A}^{(1)}$ and $f_0 \in \widetilde{\mathbf{A}}^{(n)}$.

Main ideas

As $g \in \mathbf{A}^{(1)}$. We have, by **Fact 8**,

$$g(x) = \lambda x + g(\theta).$$

We can show that $\lambda = 1$, so that

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Hence

$$f(x_1, \cdots, x_n) = f_0(x_1, \cdots, x_n) + a$$

where $a \in A_0$. Since $f_0 \in \widetilde{\mathbf{A}}^{(n)}$, we have, by **Fact 7**, that

$$f(x_1, \cdots, x_n) = \sum_{k=1}^{n} \lambda_k x_k + a$$

where $\lambda_1, \cdots, \lambda_n \in \mathcal{K}$ and $\sum_{k=1}^{n} \lambda_k = 1$.

Main ideas

(iii) If $\mathbf{A}^{(3)} = \mathbf{A}^{(3,1)}$, then by **Fact 9** $\mathbf{A}$ is the class of all term operations f :

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Let $\mathcal{G}$ be the group containing the identity transformation only and $A_0 = \emptyset$. Obviously, A_0 is normal with respect to $\mathcal{G}$.

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If $\mathbf{A}^{(1)} \neq \mathbf{A}^{(1,0)}$. Put

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Then $\mathcal{G}$ is a group with respect to the operation

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$$f(x_1, \cdots, x_n) = g(x_j) \quad (1 \leq j \leq n)$$

or

$$f(x_1, \cdots, x_n) = a,$$

where $g \in \mathcal{G}$.

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(i) $\mathbf{A}^{(n)} = \mathbf{A}^{(n,1)}$, for all $n \geq 1$.
(ii) A is an affine algebra, namely, there is a field $\mathcal{K}$ such that A is a linear space over $\mathcal{K}$ and further, there exits a linear subspace A_0 of A such that **A** is the class of all term operations f defined as

$$f(x_1, \cdots, x_n) = \sum_{k=1}^{n} \lambda_k x_k + a,$$

where $\lambda_1, \cdots, \lambda_k \in \mathcal{K}, \sum_{k=1}^{n} \lambda_k = 1$ and $a \in A_0$.

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Let $t(x_1, \cdots, x_n)$ be a truly n-ary term operation of A with $n \geq 3$. Then A must be an affine algebra, so that

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Let $s_2, \cdots, s_{n-1}$ be unary term operations of A. Then

$$s_2(x) = x + a_2, \cdots, s_{n-1}(x) = x + a_{n-1}$$

where $a_2, \cdots, a_{n-1} \in A_0$.

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$$\psi : H \longrightarrow H, u(x) \longmapsto t(x, s_2(x), \cdots, s_{n-1}(x), u(x)).$$

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Question: Can we show the map ψ is onto without using Urbanik's Representation Theorem??? :-(